Triviality of physics theorem

Hamiltonian systems:

$$\dot{x} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial x}$$

$H(x, p)$ - a C^r- smooth function, $x \in R^n$, $p \in R^n$

$\frac{d}{dt} H(x(t), p(t)) \equiv 0$ → the energy is conserved

- $n=1$ – the system is integrable:

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}(x, p(x)) \quad \text{with} \quad H(x, p(x)) = H_0$$

- $n \geq 2$ – integrability is rare

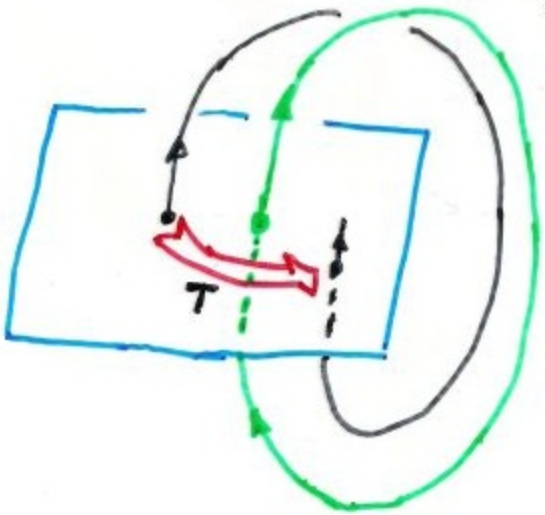
$$H(x, p) = H_1(x_1, p_1) + H_2(x_2, p_2) + \varepsilon H_3(x_1, x_2, p_1, p_2)$$

$\varepsilon = 0$: an integrable case, $\varepsilon > 0$ – chaos is possible

Q: Can we understand dynamics and Bifurcations of a given Hamiltonian system with 2 or more degress of freedom?

A: No way!

Poincare map



$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

$$H(x,p) = H_0$$

$\Downarrow$

a $(2n-1)$-dimensional system

Poincare map T is a $(2n-2)$-dimensional symplectic diffeomorphism

A map $(x,y) \mapsto (\bar{x},\bar{y})$ is symplectic if it preserves the standard sympletic form $\sum_{i=1}^{2n-2} dx \wedge dy_i$:

$$\begin{pmatrix} \frac{\partial \bar{x}}{\partial x} & \frac{\partial \bar{x}}{\partial y} \\ \frac{\partial \bar{y}}{\partial x} & \frac{\partial \bar{y}}{\partial y} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\partial \bar{y}}{\partial y} & -\frac{\partial \bar{y}}{\partial x} \\ -\frac{\partial \bar{x}}{\partial y} & \frac{\partial \bar{x}}{\partial x} \end{pmatrix}^{T}$$

Theorem. Arbitrarily close (in C^r-topology, any r) to any symplectic map with an elliptic periodic orbit there are universal symplectic maps.

Any Hamiltonian system with an elliptic periodic orbit is „pregnant" By the whole of symplectic dynamics

Universal maps

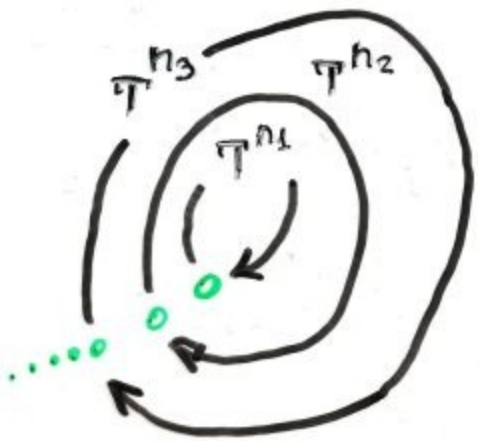
Th. Let a symplectic map T of R^{2m} have an elliptic periodic orbit. Then, c^r-generically (any r) T is universal

Def. We call T universal if its renormalization class is dense among all symplectic c^r-diffeomorphisms $D^{2m} \to R^{2m}$

The renormalization class of T =

$$= \bigcup_{n \geq 1, \text{all } \Psi} \Psi^{-1} \circ T^n \circ \Psi$$

Ψ: a symplectic diffeomorphism times a constant factor (scaling)



Elliptic fixed point

$\bar{z} = f(z)$ – a symplectic map in R^{2m}

$z_0 = f(z_0) \Rightarrow$ z_0 is a fixed point

$$\bar{z} - z_0 = f'(z_0)(z - z_0) + O(z - z_0)$$

z_0 is elliptic, if all eigenvalues of $f'(z_0)$ lie on the unit circle: $e^{\pm i\omega_1}, \dots, e^{\pm i\omega_m}$

By symplecticity, if λ is an eigenvalue, λ^{-1} is an eigenvalue too

Generically, $\{\omega_1, \dots, \omega_m, \pi\}$ are rationally independent.

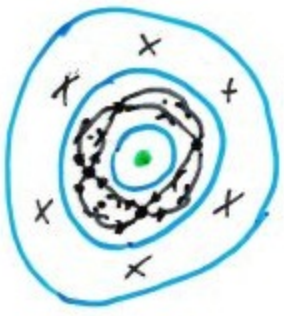
There exists then a symplectic coordinate transformation which brings the map to the form

$$\boxed{\begin{aligned} \bar{\rho} &= \rho \\ \bar{\varphi} &= \varphi + \Omega(\rho) \end{aligned}} + O(\rho^n)$$

$\rho_1 \geq 0, \dots, \rho_m \geq 0$

$(\varphi_1, \dots, \varphi_m) \in T^m$

$\Omega(0) = (\omega_1, \dots, \omega_m)$



Elliptic point is surrounded by **KAM-tori** and resonances

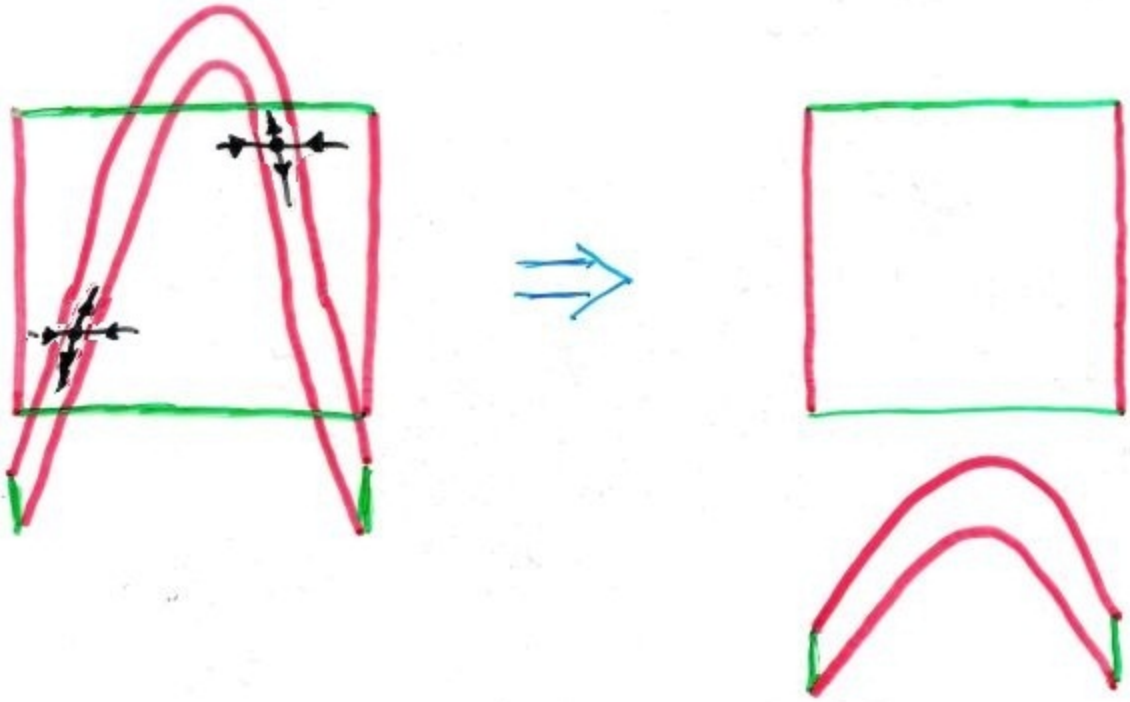
Chaos near resonances can be ultimately rich

Examples:

1) $$H = \frac{1}{2}(p_1^2 + \dots + p_n^2) + V(x_1, \dots, x_n)$$

Elliptic periodic orbits exist near a stable equilibrium (minimum of V)

2) Bifurcations within chaos: two-dimensional case



When the horseshoe unravels,

the two fixed points must collide $\Rightarrow$ near the moment of the collision one of the points must be elliptic

3) In the multidimensional case the uniform partial hyperbolicity is the only known obstacle to the existence of elliptic orbits

Claim: Every symplectic dynamics can be obtained by a small perturbation near elliptic point

Proof: **Step 1.** Normal form near the elliptic point: if $\omega_1, \dots, \omega_m, \pi$ are rationally independent, then

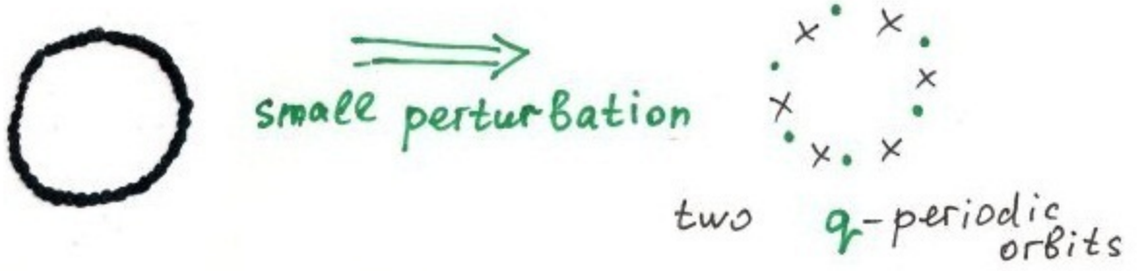
$$\boxed{\begin{aligned}\bar\rho &= \rho \\ \bar\varphi &= \varphi + \Omega(\rho)\end{aligned}} + o(\rho^r) \qquad \begin{aligned}&\rho_1 \geq 0, \dots, \rho_m \geq 0 \\ &(\varphi_1, \dots, \varphi_m) \in T^m \\ &\Omega(0) = (\omega_1, \dots, \omega_m)\end{aligned}$$

$\Downarrow$

the normal form approximates the map locally arbitrarily well in the C^r-topology

A resonance: ρ_0 such that $\Omega(\rho_0) = (2\pi\frac{p_1}{q_1}, 2\pi\frac{p_2}{q_2}, \dots, 2\pi\frac{p_m}{q_m})$

$\rho = \rho_0$ is a torus filled by q-periodic orbits



The q-th iteration of the normal form:

$$\underbrace{\bar\rho = \rho, \ \bar\varphi = \varphi + q(\Omega(\rho) - \Omega(\rho_0))}$$

$\downarrow$

the time-1 shift by the Hamiltonian flow

$$H(\rho, \varphi) = -\phi(\rho), \quad \nabla\phi = q(\Omega(\rho) - \Omega(\rho_0))$$

Step 2 Perturbation near resonance

$$H = -\Phi(\rho) + \underbrace{\varepsilon}_{\text{small}} V(\rho, \varphi)$$

A model case:

$$H = \frac{\rho^2}{2} + \varepsilon V(\varphi) \rightarrow \ddot{\varphi} + \varepsilon V'(\varphi) = 0$$

$\rho \geq 0$, $\varphi \in S^1$

The general case V can be chosen such that near the elliptic point (the newborn one)

$$\ddot{\varphi}_1 + \varepsilon(\varphi_1 - \varphi_1^*) = 0$$

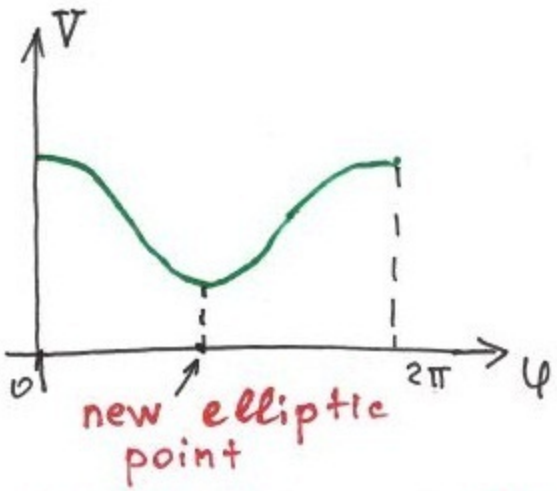
$$\vdots$$

$$\ddot{\varphi}_m + \varepsilon(\varphi_m - \varphi_m^*) = 0$$

locally, linear rotation

$\Downarrow$

some iteration of the map is identity



strictly linear oscillations if V is strictly quadratic in a neighbourhood of its minimum

Morse Lemma: locally

$$\Phi(\rho) = \frac{1}{2} p^T Q p \qquad P = \eta(\rho)$$

$$Q = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \cdots & \lambda_m \end{pmatrix} \qquad \Psi = (\eta'(\rho)^T)^{-1}(\varphi - \varphi^*)$$

locally $V = -\sum_{i=1}^{m} \lambda_i \Psi_i^2$

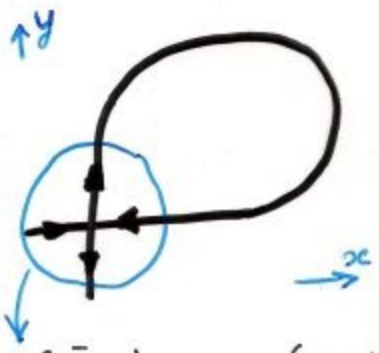
Step 3. Perturbation of identity

Time - δ map of a Hamiltonian flow is an $O(\delta)$-perturbation of identity

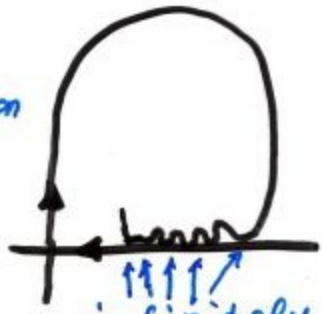
$$H(x_1, \dots, x_m; p_1, \dots p_m) = \hat{H}(x_1, p_1) + \hat{H}(x_2, p_2) + \dots + \hat{H}(x_m, p_m)$$

$$\hat{H} = \frac{p^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$

Step 4. Unfolding the homoclinic tangency



small perturbation ⟹

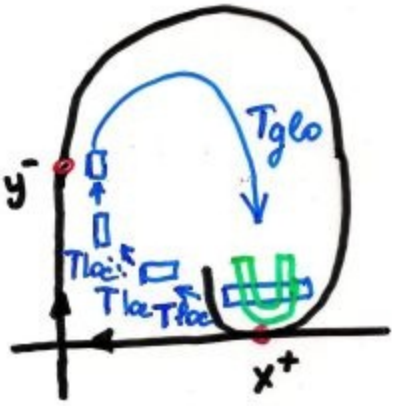


infinitely many tangencies of all orders

$$\begin{cases} \bar{x} = \lambda x + o(x,y) \\ \bar{y} = \lambda^{-1} y + o(x,y) \end{cases}$$

$0 < \lambda < 1$

Compute the first return map near a homoclinic tangency of order K



$$T_{loc}^{j}: \begin{cases} x' = \lambda^j x + o(\lambda^j) \\ y' = \lambda^{-j} y + o(\lambda^j) \end{cases}$$

$$T_{glo}: \begin{cases} \bar{x} = x^+ + (y' - y^-) + \dots \\ \bar{y} = \frac{\partial}{\partial y}\left(\mu_1 (y'-y^-) + \dots + \mu_K (y'-y^-)^K + O(\|y'-y^-\|^{K+1})\right) - x' + \dots \end{cases}$$

small parameters

$$x = x^+ + \lambda^{\frac{j}{K-1/2}} u, \quad y = \lambda^j \left[y^- + \lambda^{\frac{j}{K-1/2}} V \right]$$

$$T_{glo} \cdot T_{loc}^{K}: \begin{cases} \bar{u} = V \\ \bar{v} = \frac{\partial}{\partial v}(M_1 V + \dots + M_K V^K) - u + o(1) \end{cases} \quad j \to \infty$$

rescaled μ's

M's are no longer small!

$$\begin{cases} \bar{u} = V \\ \bar{v} = -u + \nabla \Psi(V) \end{cases}$$

Ψ: an arbitrary polynomial function

Step 5. Symplectic polynomial approximation

Theorem (Turaev, 2003):

Any symplectic diffeomorphism $D^{2m} \to R^{2m}$ can be approximated by a polynomial symplectic map with a polynomial inverse:

Not obvious: $T: \begin{cases} \bar{x} = f(x,y) \\ \bar{y} = g(x,y) \end{cases}$

is area-preserving iff

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \equiv 1$$

The approximating polynomial is found in the form $T_n \circ \dots \circ T_1$

where $T_k = \begin{cases} \bar{x} = y \\ \bar{y} = -x + \nabla \Phi_k(y) \end{cases}$

first-return maps near the chain of tangencies !

Physics is easy:

Anything is good for everything:

1. Take any physical process
2. Determine the number of degrees of freedom
3. Take any Hamiltonian with this number of degrees of freedom
4. You have an accurate description of the process under study

(just choose variables right and tune the parameters)

Symplectic polynomial approximation

Area-preserving diffeomorphism g is the time-1 shift of a Hamiltonian flow

$$\begin{cases} \dot{x} = \frac{\partial H(x,y,t)}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x}(x,y,t) \end{cases}$$

Hence $g = g_m \circ \dots \circ g_1$ where

$$g_i = \begin{cases} \bar{x} = x + \tau \frac{\partial H}{\partial y}(x,y,i\tau) + O(\tau^2) \\ \bar{y} = y - \tau \frac{\partial H}{\partial x}(x,y,i\tau) + O(\tau^2) \end{cases}$$

$$\left(\tau = \frac{1}{m}\right)$$

Uniformly with respect to $\theta \equiv i\tau$ this map is $O(\tau^2)$-close to the time-2π map by the flow

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \tau V_x'(x,t;\theta) \end{cases}$$

where $H(x,y;\theta) = \int_0^{2\pi} V(x\cos s + y \sin s, s; \theta)\, ds$

The map q defined By

$$\begin{cases} \dot{x} = y \\ \dot{y} = W(x,t) \end{cases}$$

satisfies $q = q_m \circ \dots \circ q_1$ where

$$q_i : \begin{cases} \bar{x} = x + \tau y + O(\tau^2) \\ \bar{y} = y + \tau W(x, i\tau) + O(\tau^2) \end{cases}$$

$$\left(m = \frac{1}{\tau}\right)$$

$$q_i = R_4 \circ R_3 \circ R_2 \circ R_1 + O(\tau^2)$$

$$R_1 : \begin{cases} x_1 = y \\ y_1 = -x \end{cases}$$

$$R_2 : \begin{cases} x_2 = y_1 \\ y_2 = -x_1 \end{cases} \quad \left(\begin{matrix} = -x \\ = -y \end{matrix}\right)$$

$$R_3 : \begin{cases} x_3 = y_2 \\ y_3 = -x_2 - \tau y_2 \end{cases} \left(\begin{matrix} = -y \\ = x + \tau y \end{matrix}\right)$$

$$R_4 : \begin{cases} x_4 = y_3 \\ y_4 = -x_3 + \tau W(y_3; \theta) \end{cases} \left(\begin{matrix} = x + \tau y \\ = y + \tau W(x;\theta) + O(\tau^2) \end{matrix}\right)$$

$$\theta = i\tau$$

Stickiness and flicker-noise

Hamiltonian dynamics with 2 degrees of freedom:

$x = (x_1, x_2)$, $p = (p_1, p_2)$

Energy $H(x, p)$:

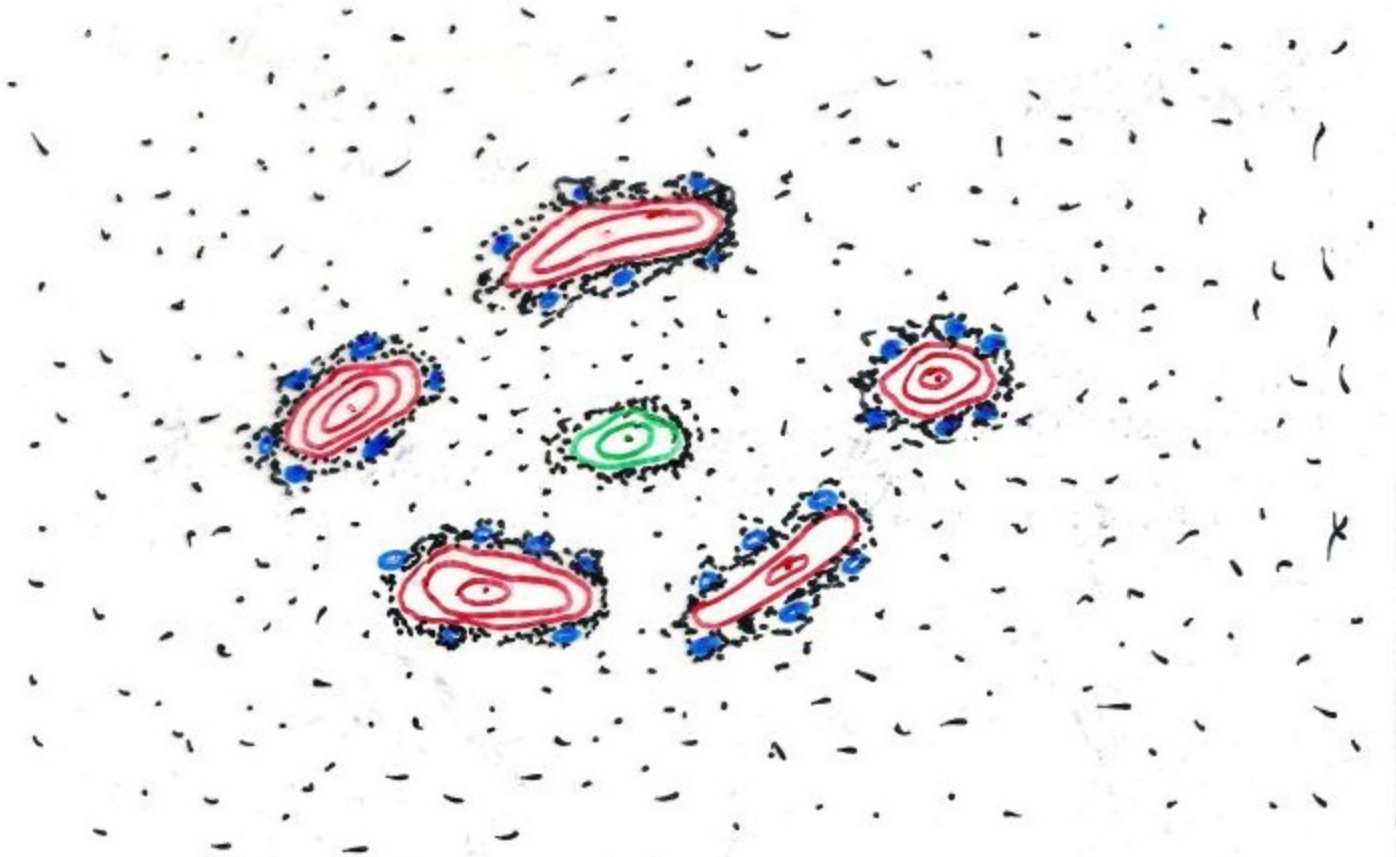
$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

The energy is conserved: $\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt} = 0$

The energy level $H(x, p) = \text{const}$ is a 3-dimensional invariant manifold $\Rightarrow$ the Poincare map is a 2-dimensional area-preserving map and the dynamics is roughly like that:

we usually see a chaotic sea with elliptic islands and the islands are sticky

(Zaslavsky, Shepelyansky, Chirikov)



NoBody believes that the physical measure sits on the boundaries of islands

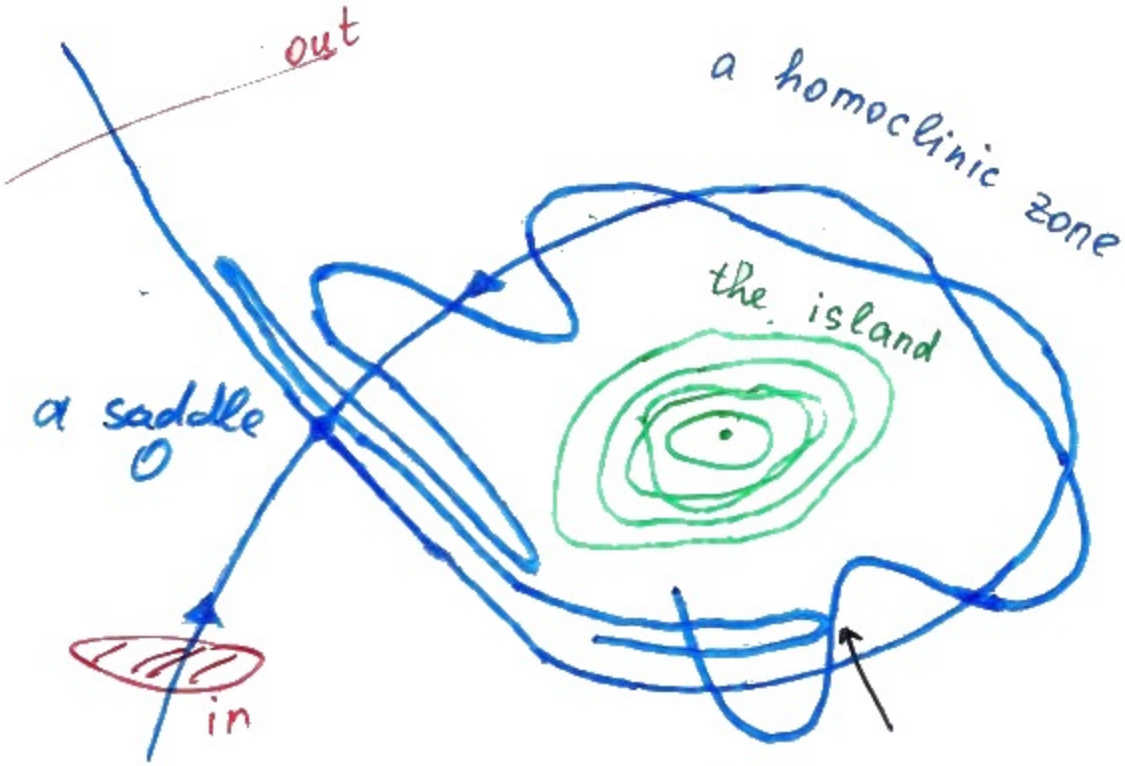
So why are the islands sticky?

the limit of iterations of an initial measure

Numerical experiments suggest that convergence of the iterations of an initial measure to the Lebesque measure slows down near the islands boundaries

The reason: once a phase point gets close to the islands it takes a long time to get out

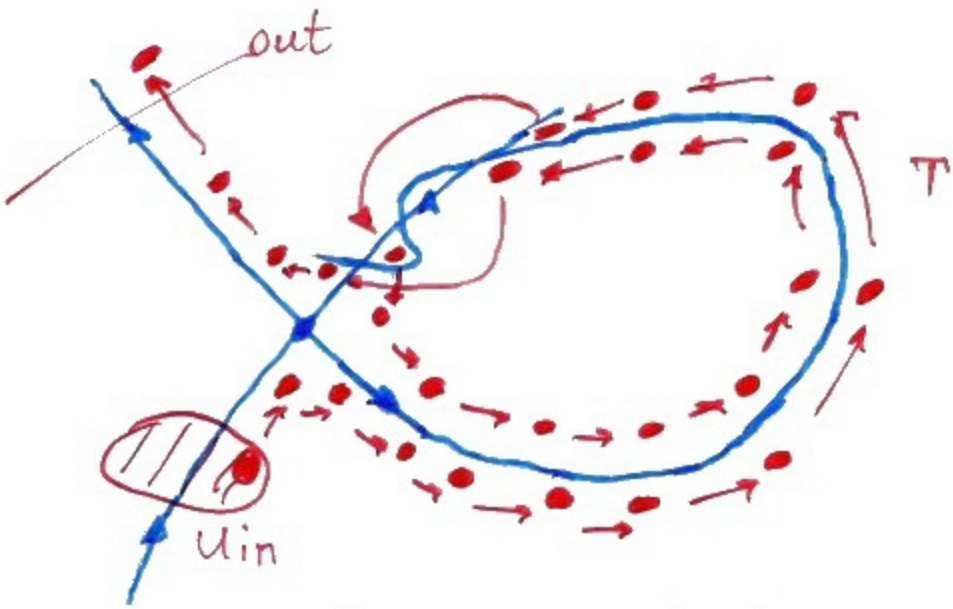
How long does it take to pass an island by?



Assumption:
the map belongs to the Newhouse domain

Standard conjecture:
every such map is in the Newhouse domain

by an arbitrarily small perturbation the saddle O will have a homoclinic tangency



$U_{in} = \bigcup_{j=1}^{\infty} U_j \Leftarrow$ the set of points which need exactly j iteration to get out

$$T^k(U_j) \cap T^s(U_i) = \emptyset \quad (\text{if } i \neq j \text{ or } s \neq k)$$
$$0 \leq k \leq j \qquad 0 \leq s \leq i$$

$$\Downarrow$$

$$S(T^k U_j) = S(U_j)$$

$$\Downarrow$$

$$\sum_{j=1}^{\infty} j\, S(U_j) < \infty$$

Probability of the residence time

$$j = P_j = \frac{S(U_j)}{S(U_{in})}$$

Average residence time $E\tau = \sum_{j=1}^{\infty} j P_i$ is always finite

Theorem For any <u>generic</u> map T from the Newhouse domain

$$E\tau^{1+d} = +\infty \qquad (*)$$

for every $d > 0$

$$\left(E\tau^{1+d} = \sum_{j=1}^{\infty} j^{1+d} p_j\right)$$

i.e. $p_j > \frac{1}{j^{2+d}}$

Proof. Enough to prove that given any T from the Newhouse domain $\mathcal{D}$ (in the space of real-analytic area-preserving maps) and given any $d > 0$, arbitrarily close to T there exists $\widetilde{T}$ such that

$$E\tau^{1+d} = +\infty \text{ for the map } \widetilde{T}$$

in other words, the sets (open)

$$\mathcal{D}_{n,N} = \left\{\text{the set of maps for wich } E\tau^{1+\frac{1}{n}} > N\right\}$$

are each dense in $\mathcal{D}$, so each map from $\mathcal{D}^* = \bigcap_{n,N=1}^{\infty} \mathcal{D}_{n,N}$ satisfies

$$E\tau^{1+d} = +\infty \qquad \text{for all } d > 0$$

Proof (continuation)

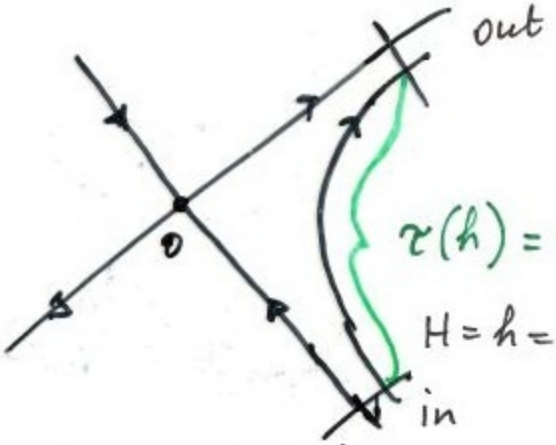
How to make $E\tau^{1+\alpha}$ infinite?

Easy, if a sufficiently degenerate saddle can be created

Example:

$$H = \frac{1}{2n}\left(x^{2n} - y^{2n}\right)$$

$$\dot{x} = y^{2n-1}, \quad \dot{y} = x^{2n-1}$$



$\tau(h)$ = the flight time

$H = h = \text{const}$

$$\frac{dx}{dt} = \left(x^{2n} - 2nh\right)^{1-\frac{1}{2n}} \Rightarrow \tau(h) = \int \frac{dx}{\left(x^{2n} - 2nh\right)^{1-\frac{1}{2n}}}$$

scaling: $x = h^{\frac{1}{2n}} z$

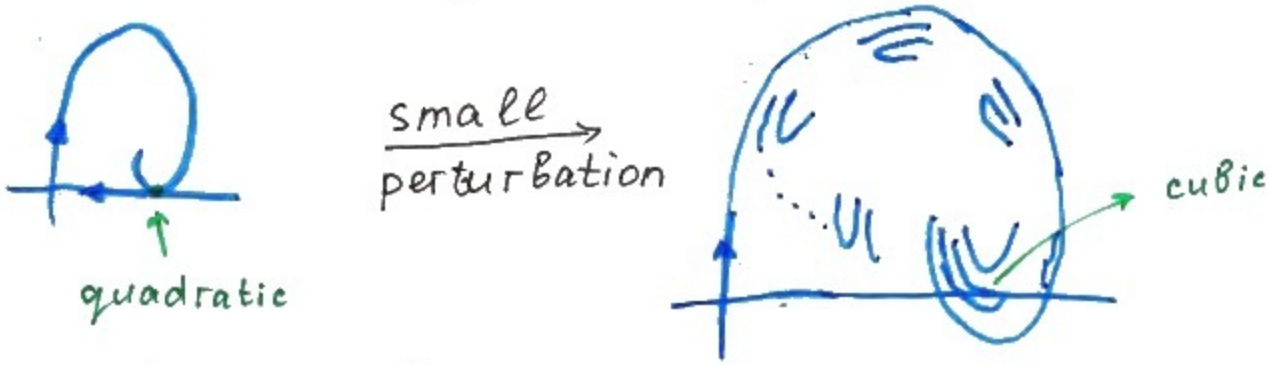
$$\tau(h) \sim h^{\frac{1}{n}-1}$$

$$E\tau^{1+\alpha} \sim \int_0^{h_0} \tau^{1+\alpha}(h)\,dh \sim \int h^{\left(\frac{1}{n}-1\right)(1+\alpha)}\,dh = \infty$$

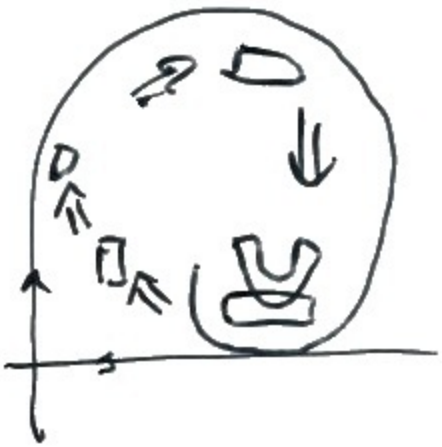
if $\alpha \geq \frac{1}{n-1}$

The higher degeneracy (n), the lower momenta will diverge

Perturbation of a quadratic tangency can create tangencies of any order:



Perturbation of an order-n tangency creates degenerate saddles of order n



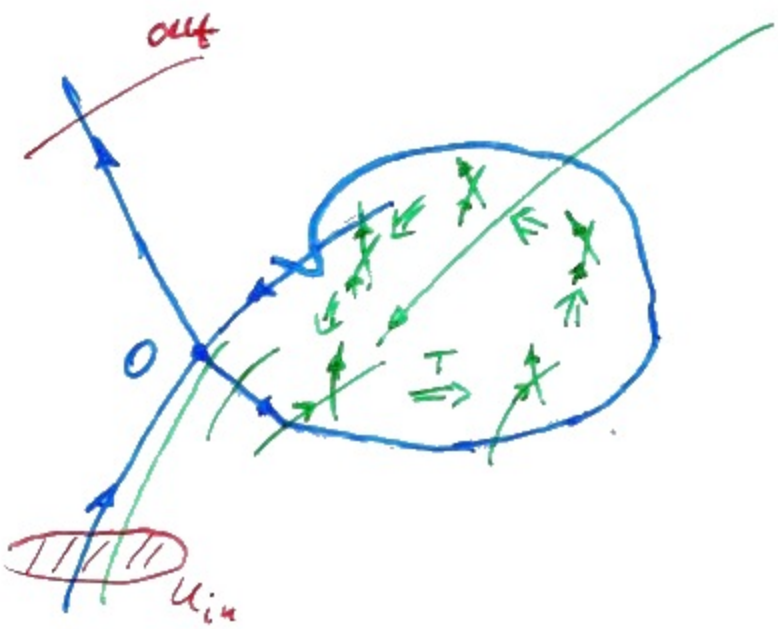
first-return map (in rescaled coordinates)

$$\begin{cases} \bar{x} = y \\ \bar{y} = \sum_{i=0}^{n-1} M_i y + y^{n+1} - x \end{cases}$$

Gonchenko, Turaev, Shilnikov (Nonlinearity, 2007)

Saddle periodic orbits of arbitrarily high order of degeneracy can be created by arbitrarily small perturbation of any map from the Newhouse domain

With some extra work: the newborn degenerate saddles have heteroclinic connections with the original saddle point O, hence their separatrices intersect U_{in}



End of proof.

Numerical evidence
(Shepelyansky)

$\psi(t)$ = the probability density for the trapping time t

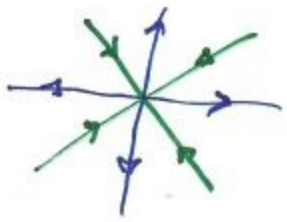
$\sim t^{-3/2}$

(10^{13} iterations of the standard map)

$$\begin{cases} \bar{x} = x + y \\ \bar{y} = y + K \sin(x+y) \end{cases}$$

$$E\, t^{3/2} = \infty$$

My interpretation:
the most degenerate saddle in 1-parameter families of area-preserving maps is the strong resonance 1:4



This gives exactly

$$E\, t^{3/2} = \infty$$

Flicker - noise

$\bar{J}$ = averaged (over time) electric current

$\Delta J = J(t) - \bar{J}$ ← fluctuations

$$\langle \Delta J(t) \cdot \Delta J(t+\tau) \rangle = K(\tau)$$

↖ time average ↗ ↖ correlation function

$$P(\omega) = \int_0^{+\infty} K(\tau) \cos \omega\tau \, d\tau = \langle \Delta J_\omega^2 \rangle$$

↖ power spectrum

Experimental observation (low temperatures)

$$P(\omega) \sim \frac{\bar{J}^2}{\omega \cdot n}, \quad 1_{month}^{-1} < \omega < 1_c^{-1}$$

↑ the number of carriers in the sample

$P \sim \frac{1}{\omega}$ = very slow decay of correlation

$P \sim \frac{\bar{J}^2}{n}$ = each particle fluctuates independently

$J = J_1 + J_2 + \ldots + J_n \Rightarrow \bar{J} = n\bar{j} \Rightarrow P \sim n$

$$\langle (\Delta J_{1,\omega} + \ldots + \Delta J_{n,\omega})^2 \rangle = \sum_{m,l=1}^{n} \langle \Delta J_{m,\omega} \Delta J_{l,\omega} \rangle = n \langle \Delta J_\omega^2 \rangle$$

Usual interpretation:
anomalously long trapping times

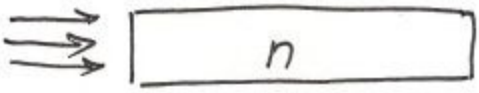
$J = e v n \rightarrow$ number of carriers

e: electric charge; v: average velocity

At low temperature

$$J(t) = e\bar{v}\, n(t)$$

$$\bar{J} = e\bar{v}\,\bar{n} \qquad \Delta J = e\bar{v}\,\Delta n$$



If there is a constant flux of non-interacting particles to the sample, then the correlation function for $\Delta n(t)$ is given by

$$K(\tau) = \bar{n}\,\frac{\int_0^{+\infty} (1 - F(t+\tau))\, F(t)\, dt}{\int_0^{+\infty} t\, \psi(t)\, dt}$$

Where $F(t)$ is the distribution function for the residence time

$$F(t) = \int_0^t \psi(s)\, ds =$$ the probability for an injected particle to stay in the sample for the time less than t

$\psi(s)$: probability density

It is legitimate now to assume that in a generic sample the statistics for residence times is such that

$$E\,t < \infty \quad , \quad E\,t^{1+\alpha} = \infty \qquad (\alpha > 0)$$

$$\Downarrow$$

$$\int_0^\infty t\,\psi(t)\,dt < \infty \quad , \quad \int_0^\infty t^{1+\alpha}\,\psi(t)\,dt = \infty$$

$$\psi(t) \sim \frac{1}{t^{2+\varepsilon}}$$

$$\Downarrow$$

$$F(t) \sim 1 - \frac{1}{t^{1+\varepsilon}}$$

$$K(\tau) \sim \int_0^{+\infty} (1 - F(t+\tau))\,F(t)\,dt \sim \int \frac{dt}{(t+\tau)^{1+\varepsilon}}$$

$$\Downarrow$$

$$K(\tau) \sim \frac{1}{\tau^{\varepsilon}}$$

$$P(\omega) \sim \int_0^{+\infty} \frac{\cos \omega\tau}{\tau^{\varepsilon}}\,d\tau \sim \frac{1}{\omega^{1-\varepsilon}}$$

$$\Downarrow$$

for arbitrarily small ε